Different equilibrium paths related to several values of the coupling parameters b and h are plotted in Fig. 4. At the value of $b = 0$, a bifurcation point exists corresponding to a new path start. When $b \neq 0$ the bifurcation point disappears and the equilibrium paths related to different values of the h parameter become asymptotic to that related to $b = 0$. For example, the path related to the $b = 0.1$, $h = 0.01$ values presents an inflection point, whereas for the $b = 0.1$, $h = 1$ value, the bifurcation point becomes a limit point and the snap through exists. By comparing Fig. 4 to Fig. 3 we can conclude that the present one-degree-of-freedom model perfectly simulates the behavior of long flat plates loaded in compression.

V. Concluding Remarks

Concerning the analysis of the nonlinear response of the cylindrical bending given by in-plane axial loads of asymmetrically laminated plates of cross-ply type, this Note has presented and compared results of three different models. The first model is an analytical one as proposed by Sun and Chin, the second consists of an FEM model, and the last one consists of a one-degree-of-freedom model. This study has confirmed that as the coupling stiffness magnitude increases, the importance of the nonlinear effects increases, and these effects cannot be neglected even when low levels of the applied load are considered. Furthermore, the following new conclusions can be noted. 1) The linearized nature of the Sun and Chin model has been shown, and its inadequacy to forecast the nonlinear response of the plates has been pointed out. 2) The FEM analysis has confirmed the preceding point, and a different level of accuracy of the analytical model between the tension and the compression loading cases, respectively, has been established. 3) The FEM analysis has shown the possibility of snapping in the nonlinear response of long plates in compression. In such a case the nonlinear effects could be dramatic. 4) The one-degree-of-freedom model has shown, in a simple manner, the conclusions given in the preceding points.

Additional investigations directed toward relating the snapping to some mixed geometrical stiffness parameter of the plate are needed. These studies should take into account different geometrical and mechanical boundary conditions. Furthermore, a mathematical treatment addressed to explain the failure of the analytical model would be welcome, which could be a subject for future work.

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Improved Method for Evaluating Damping Ratios of a Vibrating System

M. Liu*
Jilin University of Technology, Changchun 130022, People's Republic of China
and
D. G. Gorman†
Robert Gordon Institute of Technology, Schoolhill, Aberdeen AB9 1FR, Scotland, United Kingdom

Nomenclature

$[A]^\alpha$	= αth order spectrum factorization to matrix A
$[C]$	= damping matrix
$[\bar{C}]$	= modal transformed damping matrix
$\bar{C}_{\alpha\beta}$	= partitions of modal transformed damping matrix, $\alpha = k, d$, $\beta = k,d$
$[C_{cr}][D_{cr}]$	= critical damping matrices
$[D_{cr}]_k$	= truncated critical damping matrix associated with kept modes
$[I]$	= identity matrix
$[K]$	= stiffness matrix
$[M]$	= mass matrix
$[X]$	= general displacement vector
$[\Phi]$, $[\Psi]$	= right and left eigenvector matrices
$[\Phi_q]$, $[\Psi_q]$	= transformed right and left eigenvector matrices
$\Phi_k, \Phi_d, \Psi_k, \Psi_d$	= kept and deleted eigenvector matrices
Φ_r, Ψ_r	= residual right and left modes
$[\Lambda]$	= diagonal eigenvalue matrix
Λ_k, Λ_d	= kept and deleted eigenvalue matrices
$[\Lambda_r]$	= matrix obtained by residual mode transformation
α_i, β_i	= eigenvalues
λ	= scalar or eigenvalue
$[\xi]$	= Inman's criterion matrix
ζ_i, ξ_{ik}	= eigenvalues of $[D_{cr}]$ and $[D_{cr}]_k$

Introduction

IN the analysis of a dynamic system, inclusion of a damping matrix is almost unavoidable since it can often have a critical bearing on stability and dynamic responses. Damping ratios derived to classify modes of vibration determine the

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*Currently Associate, Petroleum Science and Technology Institute, Robert Gordon Institute of Technology, Schoolhill, Aberdeen AB9 1FR, Scotland, United Kingdom.

†Professor and Head, School of Mechanical and Offshore Engineering.

nature of the system. Because of the complexity of damping mechanisms, it would appear that the damping matrix cannot be explicitly formulated in most practical cases. Commonly, it does not satisfy Caughey's condition,[1] although Rayleigh's damping is assumed in many analyses. As a result of this, the damping matrix can seldom be diagonalized by the real mode matrix of vibration, and damping ratios are not readily obtainable. A complex eigenvalue solution may be performed by arranging the normal equations into state space using a state variable. However, such an augmentation leads to a non-positive definite matrix eigenproblem with a doubled number of degrees of freedom. As such, the solution process needs more computer time and more memory capacity resources. Therefore, in practice, the use of this scheme is often prohibitive for problems consisting of a large number of degrees of freedom as, for example, with the finite element method.

In view of the above, alternatives were proposed by Inman and others.[2-6] Inman and Andry firstly developed a theory dealing with damped systems by examining coefficient matrices.[2] Associated criteria were presented to determine the oscillatory behavior of a symmetric system. It has been shown that, similar to a single degree-of-freedom system, the oscillatory nature of a symmetric system can be predicted by forming a critical damping matrix and studying the definiteness of a derived matrix.

Further effort was invested by Inman[3] and Ahmadian and Inman[5] to extend the theory without the symmetry restrictions to a system. A modified method, which is easy to exercise in practice, was introduced by Inman and Jiang.[6]

This Note aims to simplify the above method and to extend it to nonsymmetric systems. In addition, mode truncation is also addressed.

Theoretical Background

In consideration of difficulties experienced in calculating damping ratios of a damped system, Inman proposed a damping ratio matrix from which critical damping ratios are determined by substitution for complex eigenvalue solutions. As with a one degree-of-freedom system, the critical damping matrix $[C_{cr1}]$ is defined as

$$[C_{cr1}] = 2[M]^{1/2}([M] - 1/2[K][M]^{-1/2})^{1/2}[M]^{1/2} \tag{1}$$

Therefore, modes of vibration are classified into overdamped, underdamped,and critically damped, depending on the eigenvalues of matrix $[D_1] = [C] - [C_{cr1}]$. The criteria are obvious:

If $[D_1] = 0$, the system is critically damped in each mode.

If $[D_1]$ is positive definite, the system is overdamped in each mode.

If $[D_1]$ is negative definite, the system is underdamped in each mode.

If $[D_1]$ is indefinite, the system is mixed-damped, i.e., has both oscillatory and non-oscillatory modes.

The critical damping matrix was further modified by Inman and Jiang[6] as $[D_2] = [C_{cr2}] - [I]$, where

$$[C_{cr2}] = [C_{cr1}]^{-1/2}[C][C_{cr1}]^{-1/2} \tag{2}$$

Accordingly, the criteria were modified by determining $\beta_i = \alpha_i - 1$, where α_i are spectrum of matrix $[C_{cr2}]$. The overdamped, critically damped, and underdamped modes correspond to β_i being positive, zero, and negative, respectively. This form virtually translates the matrix criteria into scalars.

These two kinds of criteria, aimed at reducing the computing time by avoiding a complex eigenvalue solution, involve such manipulations as follows:

1) Forming

$$[\tilde{K}] = [M]^{-1/2}[K][M]^{-1/2}$$

2) Factorizing

$$[\tilde{C}_{cr}] = 2[\tilde{K}]^{1/2} = 2[\Phi][\Lambda]^{1/2}[\Phi]^T$$

where Φ and $[\Lambda]$ are the eigenvector matrix and the diagonal eigenvalue matrix of $[\tilde{K}]$, respectively.

3) Calculating

$$[C_{cr}] = [M]^{1/2}[\tilde{C}_{cr}][M]^{1/2}$$

4) Calculating

$$[C_{cr}]^{-1/2} = [\Phi][\Lambda^{-1/2}][\Phi]^T$$

where Φ is the eigenvector matrix of $[C_{cr}]$ and $[\Lambda]$ is the diagonal matrix of eigenvalues of $[C_{cr}]$.

5) Forming

$$[\xi] = [C_{cr}]^{-1/2}[C][C_{cr}]^{-1/2} - [I]$$

and calculating its eigenvalues $[\Lambda] = \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n)$. The value of α_i determines the nature of the ith mode of vibration.

The first step is trivial if $[M]$ is diagonal.

It is apparent that at least four eigensolutions are needed to reach the final step. The approach would also seem to be open to justification as to how to identify the mode order to which the obtained damping ratio belongs, since α_i are not sequenced uniquely.

With reference to the literature mentioned above, as will be discussed later, when a large number of degrees of freedom are involved in the analysis, the approach will not be practical; and the total number of calculations is not substantially smaller than performing a complex eigenvalue solution, particularly for unsymmetrical systems.

Simplification and extension are therefore proposed in this Note by using a modal transformation procedure.

Modal Transformation Method

A linear dynamic system bears the following second-order differential equations:

$$[M]\{\ddot{X}\} + [C]\{\dot{X}\} + [K]\{X\} = 0 \tag{3}$$

where $[M]$, $[C]$, and $[K]$ are mass, damping, stiffness matrices and $\{X\}$ is displacement vector, respectively. As commonly assumed, $[M]$ is positive definite, and [K] and $[C]$ are nonnegative definite. In a general case, they are not limited to be symmetrical. In vehicle dynamics and fluid-structure interaction analysis, governing equations are established to be unsymmetrical. In this Note, symmetry is not imposed to the system by using the left and right eigenvectors. The symmetrization of these matrices was extensively studied by Inman.[1] As assumed in Ref. 5, the discussed system is symmetrizable and, thus, possesses a full set of eigenvectors to ensure that factorization encountered in references and in this context is of meaning.

The modal equation without damping present takes the form

$$[K]\{X\} = \lambda[M]\{X\}$$

or, in the matrix form,

$$[K][\Phi] - [M][\Phi][\Lambda] = 0 \tag{4}$$

where Φ is the right eigenvector matrix and $[\Lambda]$ is the eigenvalue matrix in diagonal form because the system has a full set of eigenvectors. In conjunction with use of left eigenvector matrix $[\Psi]$, which is obtained by solving the transposed $[K]$ and $[M]$ in Eq. (3), and degenerates to the right eigenvector matrix if the discussed system is symmetrical, Eq. (4) can be transformed to

$$[\Psi]^T[K][\Phi] - [\Psi]^T[M][\Phi][\Lambda] = 0 \tag{5}$$

where the right and left eigenvector matrices are mass-weighted orthogonal. Factoring the mass matrix into such a form, $[M] = [\Psi_m]^T[\Lambda_m][\Phi_m]$, we obtain $[M]^\alpha = [\Psi_m]^T[\Lambda_m^\alpha][\Phi_m]$ (α is a real number).

Employing the transformation $[\Phi] = [M]^{-1/2}[\Phi_q]$ and $[\Psi] = [M]^{-1/2}[\Psi_q]$ yields

$$[\Psi_q]^T([M]^{-1/2})^T[K]\,[M]^{-1/2}[\Phi_q] = [\Lambda] \tag{6}$$

and $[\Psi_q]^T[\Phi_q] = I$.

Utilizing the second foregoing relation, Eq. (5) can be rewritten as

$$([M]^{-1/2})^T[K][M]^{-1/2} = [\Phi_q][\Lambda][\Psi_q]^T \tag{7}$$

This is the factored form.

Hence, as with one of the symmetrical systems, the critical damping matrix for a general unsymmetrical system can be defined as

$$[C_{cr}] = 2([M]^{1/2})^T\{([M]^{-1/2})^T[K][M]^{-1/2}\}^{1/2}[M]^{1/2} \tag{8}$$

which simplifies into

$$[C_{cr}] = 2([M]^{1/2})T[\Phi_q][\Lambda^{1/2}][\Psi_q]^T[M]^{1/2}$$

because

$$[\Phi]^{-1} = [\Phi_q]^{-1}[M]^{1/2} = [\Psi_q]^T[M]^{1/2}$$

$$[\Psi]^{-1} = [\Psi_q]^{-1}[M]^{1/2} = [\Phi_q]^T[M]^{1/2}$$

As a consequence,

$$[C_{cr}] = 2[\Psi]^{-T}[\Lambda^{1/2}][\Phi]^{-1} \tag{9}$$

The definiteness of matrix $[D] = [C] - [C_{cr}]$ governs the nature of oscillatory behavior of a system. Thus, associated with the first kind of criteria, the governing matrix is modified as

$$[D_{cr}] = \tfrac{1}{2}[\Psi]^T[D][\Phi] = \tfrac{1}{2}[\Psi]^T[C][\Phi] - [\Lambda^{1/2}]$$

$$= \tfrac{1}{2}[\bar{C}] - [\Lambda^{1/2}] \tag{10}$$

where the damping matrix noted with an overbar refers to the transformed damping matrix with right and left modal matrices.

Likewise, the second form of governing matrix takes

$$[D_{cr}] = [\xi] - I = [C_{cr}]^{-1/2}[C][C_{cr}]^{-1/2} - I$$

$$= \tfrac{1}{2}[\Lambda^{-1/4}][\bar{C}][\Lambda^{-1/4}] - I \tag{11}$$

The information of definiteness of the derived matrices allows us to use criteria summarized in the Theoretical Background Section to assess the nature of oscillatory behavior which the discussed system possesses.

It is observed that both the first and second forms are expressed in terms of modal parameters. Therefore, it requires only one modal extraction to form the governing matrices, whereas four factorizations or eigensolutions are involved in the method proposed by Inman and Jiang.

From the eigenvalues of the critical damping matrix, we know the numbers of overdamped, critically damped, and underdamped modes. It is insufficient, however, to determine the mode to which a discussed damping ratio corresponds. This issue will be discussed in the last section.

An example of a two-dimensional system is shown as follows: The mass, damping, and stiffness matrices are, respectively,

$$[M] = \begin{bmatrix} 12.50 & 8.75 \\ 5.00 & 17.50 \end{bmatrix}, \qquad [C] = \begin{bmatrix} 0.75 & 0.13 \\ -0.50 & 2.25 \end{bmatrix}$$

$$[K] = \begin{bmatrix} 500 & 250 \\ -1000 & 3500 \end{bmatrix}$$

The extracted eigenvalues and the associated left and right eigenvectors are

$$\lambda_{1,2} = 37.41669,\ 305.44$$

$$[\Psi] = \begin{bmatrix} 0.161 & 0.362 \\ 0.116 & -0.209 \end{bmatrix} \qquad [\Phi] = \begin{bmatrix} 0.227 & 0.153 \\ 0.116 & -0.209 \end{bmatrix}$$

After transformed by the left and right eigenvector matrices, the mass, damping, and stiffness matrices become

$$[\bar{M}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad [\bar{C}] = \begin{bmatrix} 0.0404 & 0.0158 \\ 0.0158 & 0.0594 \end{bmatrix}$$

$$[\bar{K}] = \begin{bmatrix} 37.41 & 0 \\ 0 & 305.44 \end{bmatrix}$$

According to Eq. (10), $[D_{cr}]$ is formed as

$$\begin{bmatrix} -6.096 & 0.008 \\ 0.008 & -17.447 \end{bmatrix}$$

with its eigenvalues being $\lambda_{1,2} = -6.683,\ -16.860$. Obviously $[D_{cr}]$ is negatively definite and the system is underdamped in each mode. The exact damping ratios of this two-dimensional system are 0.04 and 0.06. A complex eigensolution gives the following eigenvalues: $0.0245 \pm i6.117$ and $0.104 \pm i17.477$.

Modal Truncation

What is frequently encountered in engineering analysis is that only a number of modes are required in the computation. This is partially due to our interests in lower natural frequencies and for the sake of reducing computing time. In engineering problems, a finite element model consists of a large number of degrees of freedom to achieve a good approximation. Thus, an extraction of complete modes is neither practical nor necessary. Furthermore, high-order mode parameters often are not accurate, because of the approximation of finite element models and because of accumulating errors associated with iteration procedures. Typically then, we are faced with using truncated modal transformation in the method prescribed in this Note.

If the mode matrices are separated as $[\Phi] = [\Phi_k, \Phi_d]$, $[\Psi] = [\Psi_k, \Psi_d]$, and $[\Lambda] = \mathrm{diag}\{\Lambda_k, \Lambda_d\}$, the matrices in Eqs. (10) and (11) can be expressed respectively, as

$$[D_{cr}] = \tfrac{1}{2}\begin{bmatrix} \bar{C}_{kk} & \bar{C}_{kd} \\ \bar{C}_{dk} & \bar{C}_{dd} \end{bmatrix} - \begin{bmatrix} \Lambda_k & 0 \\ 0 & \Lambda_d \end{bmatrix} \tag{12}$$

and

$$[D_{cr}] = \tfrac{1}{2}\begin{bmatrix} \Lambda_k^{-1/4}\bar{C}_{kk}\Lambda_k^{-1/4} & \Lambda_k^{-1/4}\bar{C}_{kd}\Lambda_d^{-1/4} \\ \Lambda_d^{-1/4}\bar{C}_{dk}\Lambda_k^{-1/4} & \Lambda_k^{-1/4}\bar{C}_{dd}\Lambda_d^{-1/4} \end{bmatrix} - \begin{bmatrix} I_k & 0 \\ 0 & I_d \end{bmatrix} \tag{13}$$

Truncation of high-order modes means that only the top left block in each matrix is kept. Omitting the rest is equivalent to imposing restrictions to the system. In accordance with the separating eigenvalue theorem (see, e.g., Ref. 8), the eigenvalues of $[D_{cr}]_k$ fall between those of $[D_{cr}]_{K+1}$. From this recurrence relation, we get

$$\zeta_i \le \xi_i \le \zeta_{i+d}, \qquad i = 1, 2, \ldots, k \tag{14}$$

where ζ_i and ξ_{ik} are the ith eigenvalues of $[D_{cr}]$ and $[D_{cr}]_K$, respectively. From the above inequality the following conclusions can be drawn:

If $\xi_{ik} < 0 \quad (0 < i \le l)$, then $\zeta_i < 0$ and there exist l underdamped modes;

If $\xi_{ik} > 0 \quad (l < i \le k)$, then $\zeta_{i+d} > 0$ and there exist $k - l$ overdamped modes;

If $\xi_{ik} = 0$, then it is insufficient to deduce that there is a critically damped mode, i.e., $\zeta_{i+d} = 0$.

The status of the remaining d modes cannot be determined. So it is concluded that the positive or negative definiteness of the condensed matrix using modal truncation gives the same definiteness as the complete model.

It should be noted that if the damping matrix is negative or mixed definite, the system is not stable.[1] It can be deduced that if $\xi_{ik} < -1 \quad (0 < i \leq l)$, then $\zeta_i < -1$ and there exist l negative roots in the system. The system has both oscillatory and unstable modes.

Residual Modes

In modal synthesis or substructural analysis, residual modes[7] are often used in conjunction with normal modes to compensate the truncated higher-order modes so as to increase accuracy. The higher-order mode contribution is similar to static modes. On this occasion, the derived matrices, Eqs. (10) and (11), are still applicable for two reasons. Firstly, normal modes are orthogonal to the residual modes. The orthogonality can be readily examined by the residual mode definition:

$$[\Phi_r] = [\Phi_d][\Lambda^{-1}][\Psi_d]^T[0I]^T$$

$$= ([K]^{-1} - [\Phi_k][\Lambda^{-1}][\Psi_k])[0I]^T$$

So

$$[\Psi_k]^T][M][\Phi_r] = [\Psi_k]^T[M][\Phi_d][\Lambda^{-1}][\Psi_d]^T[0I]^T$$

Because the left and right eigenvectors are orthogonal in terms of the mass matrix, i.e., $[\Psi_k]^T[M][\Phi_d] = 0$. Therefore, $[\Psi_k]^T [M][\Phi_r] = 0$. Secondly, the eigenvalues are preserved by a nonsingular transformation. So, deduction follows directly from the truncated modal transformation with only an adjustment to include the residual modes. The same forms of matrices can be obtained; but one difference is that $[\Lambda_r]$ corresponding to the residual modes is not a diagonal matrix.

Identification of Mode Order

As noted earlier, there is insufficient information to determine which mode is associated with the identified damping ratio, since a matrix is conjugate with its diagonal spectrum matrix regardless of the sequence of eigenvalues. The eigenvalues of the critical damping matrix are normally sequenced from the small to the large, not according to the mode order to which they are attributed. According to the complex mode perturbation method,[8] diagonal elements of the transformed damping matrix are roughly the same as damping ratios, with an accuracy of the first order in magnitude. So, from the position of the diagonal elements, we can determine which mode a damping ratio belongs to. Or we can simply take diagonal elements as damping ratios if they are comparatively larger than off-diagonal elements.

Concluding Remarks

This Note has presented a modal transformation method for calculating a damping ratio matrix. This matrix can be used to characterize the modal behavior of dynamic systems. The method is applicable to both nonsymmetric, as well as symmetric, systems and has been shown to require fewer computational operations than similar methods in the literature. Furthermore, a modal truncation method has been discussed which is of practical use when employing finite element methods.

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Mesh Distortion Control in Shape Optimization

S. Zhang* and A.D. Belegundu†

Pennsylvania State University, University Park, Pennsylvania 16802

I. Introduction

IN structural shape optimization, one starts with some initial shape and iteratively changes it to minimize the objective function subject to constraints on structural response. However, as the shape changes, the finite element mesh can get badly distorted. Subsequent analysis is either unreliable or impossible. Distortion has been observed in many problems, particularly when optimizing a part that is overdesigned, in three-dimensional problems, and toward the end of a shape optimization run. Maintaining a good quality mesh does not explicitly enter into the problem formulation and consequently needs certain modifications.

Here, a mesh quality indicator based on the Jacobian is defined and implemented during the line search phase of the nonlinear programming algorithm. An element distortion parameter DP is first defined and a limiting value DP_l is chosen based on preventing degeneracy of the quadrilateral. A closed-form expression for the maximum step length along a known search direction is then derived.

Once the distortion parameter value reaches the limit, subsequent shape changes are arrested. Further shape improvements are possible only if the mesh is changed so that the distortion constraints are no longer active, which can be achieved by addition of elements, increasing the degree of the shape functions, or relocating the nodes. Here, a simple rezoning technique is used to relocate nodes.

II. Distortion Parameter DP

It is first necessary to define an indicator that represents the quality of a finite element mesh.

A distortion parameter for a general quadrilateral element is now introduced as[1]

$$DP = \frac{4 \det [J]_{\min}}{A} \tag{1}$$

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*Graduate Research Assistant; currently at The MacNeal-Schwendler Corporation.

†Associate Professor, Mechanical Engineering Department.